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Navier-Stokes Equations and Fluid Turbulence

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Abstract

An Eulerian-Lagrangian approach to incompressible fluids that is convenient for both analysis and physics is presented. Bounds on burning rates in combustion and heat transfer in convection are discussed, as well as results concerning spectra.

Incompressible fluids are described by the Navier-Stokes equation. Turbulence ([1], [2], [3]) experiments provide measurements that correspond to averages of certain quantities associated to the variables appearing in the Navier-Stokes equation. The present mathematical knowledge about the Navier-Stokes equations is incomplete. Some of the quantities measured in experiments are accessible to mathematical theory. They are usually low order, one-point bulk averages like the time average of integrals of squares of gradients. Most other measured quantities are not amenable to rigorous quantitative a priori analysis. Turbulence is concerned with statistical or collective properties of fluids. Nevertheless, the main impediment to progress in the rigorous analysis of turbulence is the present lack of understanding of possible blow up in individual solutions of the Euler and Navier-Stokes systems.

I will discuss briefly the blow up problem and present an Eulerian-Lagrangian approach to fluids. I will also give examples of low order one-point bulk quantities that can be treated with present knowledge and discuss results on certain two-point quantities.

1   An Eulerian-Lagrangian Approach to Fluids

I will start by recalling that the Navier-Stokes-Euler system can be written as an evolution equation for the three-component velocity vector $u = u(x, t)$,

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \Delta u + f;$$

the pressure $p = p(x, t)$ preserves incompressibility

$$\nabla \cdot u = 0.$$

The Euler system is obtained if the kinematic viscosity vanishes, $\nu = 0$; the Navier-Stokes system if $\nu > 0$. Boundary conditions are different for the two

systems. The blow up question can be stated in its simplest form for the pure initial value problem: are there any smooth initial data with finite energy that lead to solutions that diverge in finite time? The answer is not known.

For a blow up in the Navier-Stokes system one would have to have a finite time divergence of an eddy-viscosity-like quantity:

$$\int_0^T \sup_{x,r} |u(x+r,t) - u(x,t)|^2 dt < \infty \Rightarrow u \in \mathbf{C}^\infty.$$

By contrast, it is known ([4]) that

$$\int_0^T \sup_{x,r} |u(x+r,t) - u(x,t)| dt < \infty.$$

In a situation in which all velocities are finite no singularities can appear in the Navier-Stokes equations. One could accept the finiteness of velocities as a physical hypothesis. For the Euler system this hypothesis would not be sufficient to ensure smoothness of solutions ([5]). A sufficient condition for regularity in the Euler equations is the finiteness of the time integral of the maximum magnitude of vorticity ([6]). The vorticity (the curl of velocity or anti-symmetric part of the velocity gradient), $\omega = \nabla \times u$ obeys a quadratic evolution equation. The magnitude of vorticity obeys
$$D_t|\omega| = \alpha|\omega|$$
with $D_t = \partial_t + u \cdot \nabla$ the material derivative along particle paths. The scalar stretching term α is related to the magnitude of vorticity by a principal value singular integral ([7]):

$$\alpha(x) = P.V. \int D\left(\hat{y}, \xi(x), \xi(x+y)\right) |\omega(x+y)| \frac{dy}{|y|^3}$$

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}.$$

The smooth, mean zero function of three unit vectors D vanishes when two of its arguments are on the same line. Consequently, if the direction of vorticity ξ is Lipschitz then the singular integral representing α is mild and the solutions remain smooth ([8], [9], [10]). This is a generalization of the two dimensional situation where $\nabla\xi = 0$ and the solutions remain smooth. The possibility of blow up due to strong vortex stretching is not removed by the previous result; the result only precludes blow up in a smooth vortex line field. Blow up can occur also because of strain intensification (the strain matrix is the symmetric part of the gradient of vorticity). I will present now a description of the Euler equations that is convenient for analysis ([11]) and allows for a clearer geometric picture of the possible singularity formation. I will present the results only in the periodic case for simplicity of exposition. The case of decay at infinity is almost identical.

Theorem 1 *A function $u(x,t)$ solves the incompressible Euler equations if and only if it can be represented in the form $u = u_A$,*

$$u_A^i(x,t) = u_0^m(A(x,t))\frac{\partial A^m(x,t)}{\partial x_i} - \frac{\partial n_A(x,t)}{\partial x_i}$$

where $A(x,t)$ solves the active vector equation

$$\frac{\partial A(x,t)}{\partial t} + u_A(x,t) \cdot \nabla A(x,t) = 0$$

with initial data

$$A(x,0) = x.$$

u_0 is the initial velocity and $n_A(x,t)$ is determined up to additive constants by the requirement of incompressibility, $\nabla \cdot u_A = 0$.

Let us denote by

$$\mathbf{P} = \left(\mathbf{1} - \nabla\Delta^{-1}\nabla\cdot\right)$$

the Leray-Hodge projector on divergence-free vectors. The local existence theorem requires just one derivative to be Hölder continuous:

Theorem 2 *Let u_0 be a divergence free $C^{1,\epsilon}$ periodic vector valued function of three variables. There exists a time interval $[0,T]$ and a unique $C([0,T];C^{1,\epsilon})$ spatially periodic vector valued function $\delta(x,t)$ such that*

$$A(x,t) = x + \delta(x,t)$$

solves the active vector formulation of the Euler equations,

$$\frac{\partial A}{\partial t} + u_A \cdot \nabla A = 0,$$

$$u_A = \mathbf{P}\left\{(\nabla A(x,t))^* u_0(A(x,t))\right\}$$

with initial datum $A(x,0) = x$.

The proof of this result is based on an identity that removes the apparent ill-posedness, on singular integral calculus, and on the use of the method of characteristics. In the active vector formulation, the "back-to-labels" map A has conserved distribution. Its time evolution is a smooth, volume-preserving rearrangement. Singularities can occur only if the gradient map ∇A diverges rapidly in finite time:

$$\int_0^T \sup_x |\nabla A(x,t)|^2 dt < \infty \Rightarrow A \in \mathbf{C}^\infty.$$

Thus, would-be singularities are gradient singularities in a conserved quantity, similar to shocks in conservation laws, but with the significant difference

that the characteristic flow is measure preserving. The formula relating the velocity to the value of the spatial gradient at the same instance of time ([12], [13], [14], [15]) may have important mathematical consequences. On one hand, conservation of kinetic energy confers a constraint to the growth of ∇A. On the other hand, the formula suggests that near regions of high gradient of A the velocity is exceedingly high, making the regions of high gradient difficult to track and perhaps unstable. Dynamical stability or instability of blow up modalities is a difficult subject. There are obvious space and time symmetries (for instance, a minute delay of blow up), that clearly should not be categorized as instabilities. Nevertheless, even relatively simple PDEs can exhibit the coexistence of a variety of dynamical behaviors, including several stable blow up modalities, stable time independent solutions, unstable blow up modalities and dynamical connections between the unstable behaviors and the stable ones ([16]).

I will pass now from the blow up problem to some more tractable questions about average properties. One can obtain rigorous upper bounds for certain bulk averages of solutions of Navier-Stokes equations. Lower bounds are harder to obtain. Upper bounds for bulk averages of low order moments for Rayleigh-Bénard convection will be described further below; I start with a lower bound for the burning rate in a simple model of turbulent combustion.

2   Bulk Burning Rate

Mixtures of reactants may interact in a burning region that has a rather complicated spatial structure but is thin across ([17]). This reaction region moves towards the unburned reactants leaving behind the burned ones. When the reactants are carried by an ambient fluid then the burning rate is enhanced. The physical reason for the observed speed-up is believed to be that fluid advection tends to increase the area available for reaction. What characteristics of the ambient fluid flow are responsible for burning rate enhancement? The question needs first to be made precise, because the reaction region may be complicated and, in general, may move with an ill-defined velocity. An unambiguous quantity V representing the bulk burning rate is defined in ([18]) and explicit estimates of V in terms of the magnitude of the advecting velocity and the geometry of streamlines are derived. In situations where traveling waves are known to exist, V coincides with the traveling wave speed and the estimates thus provide automatically bounds for the speed of the traveling waves. The main result of ([18]) is the identification of a class of flows that are particularly effective in speeding up the bulk burning rate. The main feature of these "percolating flows" is the presence of tubes of streamlines connecting distant regions of burned and unburned material. For such flows we obtained an optimal linear enhancement bound $V \geq KU$ where U represents the magnitude of the advecting velocity and K is a proportionality factor that depends on the geometry of streamlines but not the speed of the flow. Other flows and in particular cellular flows, which have closed streamlines, on the other hand, may produce a weaker enhancement.

The instantaneous bulk burning rate is defined by the formula

$$V(t) = \int_D \frac{\partial T}{\partial t}(x, y, t) dx dy$$

where the integral extends over the spatial domain D, taken here for simplicity of exposition to be a two-dimensional strip of unit width and infinite length

$$0 \le y \le 1, \quad -\infty < x < \infty.$$

The temperature T is assumed to obey Neumann boundary conditions at the finite boundaries and to obey

$$T(-\infty, y) = 1, \quad T(\infty, y) = 0.$$

The simplified model is a passive reactive scalar with a KPP nonlinearity

$$T_t + u \cdot \nabla T - \kappa \Delta T = \frac{v_0^2}{4\kappa} T(1 - T).$$

with prescribed velocity u that satisfies

$$\int_0^1 u(x, y, t) \, dy = 0, \quad \nabla \cdot u = 0.$$

The constant v_0 represents the speed of a stable one-dimensional laminar ($u = 0$) traveling wave. We start with a very general lower bound:

Theorem 3 *For arbitrary initial data obeying*

$$0 \le T_0(x, y) \le 1.$$

one has the general lower bound

$$V(t) \ge C v_0 \left(1 - e^{-\frac{v_0^2 t}{2\kappa}} \right).$$

The proof is based on the product lemma

Lemma 2.1 *There exists a constant $C > 0$ such that*

$$0 \le T(x, y) \le 1,$$

$$T(-\infty, y) = 1, \quad T(\infty, y) = 0 \ \text{ for any } \ y \in [0, 1].$$

implies

$$\left(\int_D T(1 - T) \, dx dy \right) \left(\int_D |\nabla T|^2 \, dx dy \right) \ge C.$$

Although information about the velocity is not present in the general result, it nevertheless shows that this model does not permit quenching. Also, the general lower bound applies to the homogenized version of the equations as well.

For a very general class of velocities $u(x, y, t)$ it can be shown that the bulk burning rate may not exceed a linear bound in the amplitude of the advecting velocity. For a large class of flows we proved lower bounds on the bulk burning rate that are linear in the magnitude of advection. We denote by

$$\langle V \rangle_\tau = \frac{1}{\tau} \int_0^\tau V(t)\, dt$$

the time average of the instantaneous bulk burning rate. The main result of ([18]) is too technical to state here precisely but its meaning is that presence of coherent tubes of streamlines connecting unburned and burned regions enhances the burning rate

$$\langle V \rangle_\tau \geq KU$$

as long as the velocity spatial scales are not too small compared to the reaction length scale $\frac{\kappa}{v_0}$, and the time scale of change of the advecting velocity is not too small compared to $\tau_0 = \max[\frac{\kappa}{v_0^2}, \frac{\tilde{H}}{v_0}]$ where $\tilde{H}$ is associated to the width of the coherent tubes of streamlines. For instance, a result concerning mean zero shear flow of the form

$$u(x, y) = (u(y), 0), \quad \int_0^1 u(y)dy = 0$$

can be stated as

Theorem 4 *Let us consider an arbitrary partition of the interval $[0, 1]$ into subintervals $I_j = [c_j - h_j, c_j + h_j]$ on which $u(y)$ does not change sign. Denote by D_-, D_+ the unions of intervals I_j where $u(y) > 0$ and $u(y) < 0$ respectively. Then there exist constants $C_\pm > 0$, independent of the partition, $u(y)$, and the initial data $T_0(x, y)$, so that the average burning rate $\langle V \rangle_\tau$ satisfies the following estimate:*

$$\langle V \rangle_\tau \geq C_+ c_+ \sum_{I_j \subset D_+} \left(1 + \frac{l^2}{h_j^2}\right)^{-1} \int_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} |u(y)| dy$$

$$+ C_- c_- \sum_{I_j \subset D_-} \left(1 + \frac{l^2}{h_j^2}\right)^{-1} \int_{c_j - \frac{h_j}{2}}^{c_j + \frac{h_j}{2}} |u(y)| dy$$

for any $\tau \geq \tau_0 = max\left[\frac{\kappa}{v_0^2}, \frac{H}{v_0}\right]$. (H = 1). Here $l = \kappa/v_0$. The constants $c_\pm$ are defined by

$$c_\pm = \left(\sum_{I_j \subset D_\mp} \frac{h_j^3}{h_j^2 + l^2}\right) \left(\sum_{I_j} \frac{h_j^3}{h_j^2 + l^2}\right)^{-1}.$$

The main result of ([18]) applies to a large class of flows that are not necessarily spatially periodic, nor shears, and can have completely arbitrary features outside the tubes of streamlines. The bulk burning rate is still linear in the magnitude of the advecting velocity, no matter what kind of behavior (closed streamlines, areas of still fluid, etc.) the flow has outside the tubes. The proportionality coefficient depends on the geometry of the flow in a rather complex manner. These bounds can be extended to larger classes of chemistries. The lower bounds, however, are not yet available for models in which there is a feedback coupling of temperature on the velocity. For such models upper bounds can be derived. In the next section we will discuss upper bounds for heat transfer. We will concentrate on the simplest coupling, mediated by gravity in a Boussinesq approximation and discuss the canonical case of Rayleigh-Bénard convection.

3 Bulk Heat Transfer

The equations for Rayleigh-Bénard convection in the Boussinesq approximation are

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \sigma \Delta u + \sigma Ra \hat{e} T,$$

$$\nabla \cdot u = 0$$

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \Delta T.$$

The vertical direction $\hat{e}$ of gravity is singled out. We consider as spatial domain a box of height 1 and lateral side L. The velocity vanishes at the boundary, $T = 1$ at the bottom boundary and $T = 0$ at top. The Nusselt number is the space-time average of the flux of temperature across horizontal cross-section planes. From the equations of motion it follows that

$$\langle |\nabla T|^2 \rangle = N.$$

and also

$$\langle |\nabla u|^2 \rangle = Ra(N - 1).$$

Here $< \cdots >$ is global space-time average. The general rigorous result here is ([19], [20])

Theorem 5 *There exists an absolute constant C, independent of Rayleigh number Ra, aspect ratio L and Prandtl number σ such that*

$$N \leq 1 + C\sqrt{Ra}$$

holds for all solutions of the Boussinesq equations.

The experimental data ([21]) point however more to results of the type $N \sim Ra^{\frac{1}{3}}$ or $N \sim Ra^{\frac{2}{7}}$. The exponent $\frac{1}{3}$ can be obtained rigorously for a simplified model. Consider the infinite Prandtl number equations for rotating convection

$$(\partial_t + u \cdot \nabla) \, T = \Delta T$$

$$-\Delta u - E^{-1} v + p_x = 0$$
$$-\Delta v + E^{-1} u + p_y = 0$$
$$-\Delta w + p_z = RT.$$

$$\nabla \cdot \mathbf{u} = 0$$

with boundary conditions: $((u, v, w), \, p, \, T)$ periodic in x and y with period L; u, v, and w vanish for $z = 0, 1$, $T = 0$ at $z = 1$, $T = 1$ at $z = 0$. One can prove ([22])

Theorem 6   *There exist absolute constants $c_1, ..., c_4$ so that the Nusselt number for rotating infinite Prandtl-number convection is bounded by*

$$N - 1 \leq$$

$$\min \left\{ c_1 R^{\frac{2}{5}}; \, (c_2 E^2 + c_3 E) R^2; c_4 R^{1/3} (E^{-1} + \log_+ R)^{\frac{2}{3}} \right\}.$$

This coincides, in the limit of no rotation $E \to \infty$ with a logarithmic correction ([23]) to the $\frac{1}{3}$ exponent. The bound also shows that strong rotation $E \to 0$ stabilizes the system and that increasing rotation may result in a non-monotonic behavior of the Nusselt number, as observed in experiments.

Consider now the horizontal average $\overline{T}(z, t)$ of $T(x, y, z, t)$ and define

$$n = \left\langle |\nabla (T - \overline{T})|^2 \right\rangle.$$

Note that

$$n \leq N$$

Theorem 7   *For the full Boussinesq system*

$$N \leq 1 + c(nRa)^{\frac{1}{3}}$$

holds. For the infinite Prandtl number system

$$N \leq 1 + C \left(Ra (\log_+ Ra)^2 \sqrt{n} \right)^{\frac{2}{7}}$$

holds.

Note that, if $n \leq N$ is used then the result recovers the exponents $\frac{1}{2}$ for general Rayleigh - Bénard and logarithmically corrected $\frac{1}{3}$ for the infinite Prandtl number case. But the rigorous appearance of the exponent $\frac{2}{7}$ is perhaps not coincidental.

One of the technical ingredients for the proof of these results concerns zero order operators that are not translation invariant

$$B = \frac{\partial^2}{\partial z^2}(\Delta_{DN}^2)^{-1}\Delta_h$$

where $w = (\Delta_{DN}^2)^{-1}f$ is the solution of

$$\Delta^2 w = f$$

with horizontally periodic and vertically Dirichlet and Neumann boundary conditions $w = w' = 0$.

Theorem 8 *For any $\alpha \in (0,1)$ there exists a positive constant C_α such that every Hölder continuous function θ that is horizontally periodic and vanishes at the vertical boundaries satisfies*

$$\|B\theta\|_{L^\infty} \leq C_\alpha\|\theta\|_{L^\infty}\left(1 + \log_+\|\theta\|_{C^{0,\alpha}}\right)^2.$$

The proof of this result is based on a pointwise bound on exponential-oscillatory sums of the type:

$$K(x,z) = \sum_{k\in\mathbf{Z}^2} e^{\frac{2\pi}{L}ik\cdot x}m_k^p e^{-\epsilon m_k}$$

where $\epsilon = \epsilon(z) \geq 0$, $m_k = \frac{2\pi}{L}|k|$ and $\epsilon(z) = 0 \Rightarrow z = z_0$. The sum is singular at $z = z_0$ and the pointwise bound

$$|K(x,z)| \leq C_p\left[|x|^2 + \epsilon^2(z)\right]^{-\frac{p+2}{2}}$$

is obtained using the Poisson summation formula.

The bounds on bulk one-point quantities presented above are among the most successful areas of mathematical and experimental agreement. The reason is perhaps that the quantities involved are numbers, albeit numbers depending on a parameter. The next step beyond the description of bulk one-point averages is the description of power spectra. These are asserted to have some universal features in physical turbulence theory; we present some mathematical results in the next section.

4 Spectra

Unlike bulk one-point quantities, spectra are averages of functions. There are some well-established spectra in the physical literature associated to small scale turbulence: the Kraichnan spectrum in two dimensions and the Kolmogorov spectrum in three dimensions.

The energy spectrum $E(k)$ is a function that has the property that

$$\int_0^\infty E(k)dk = \left\langle|u|^2\right\rangle.$$

(The convention is that $< \cdots >$ is normalized space integral followed by long time average). The Kolmogorov spectrum for 3D turbulence is

$$E(k) = C_{Kl} \langle \epsilon \rangle^{\frac{2}{3}} k^{-\frac{5}{3}}.$$

The 2D Kraichnan spectrum is

$$E(k) = C_{Kr} \langle \eta \rangle^{\frac{2}{3}} k^{-3}.$$

Here $\epsilon = \nu |\nabla u|^2$ is the rate of dissipation of energy and $\eta = \nu |\nabla \omega|^2$ is the rate of dissipation of enstrophy. The spectra are supposed to be valid in a range of scales $k \in [k_i, k_d]$ where k_d is the dissipation scale and is determined by viscosity and ϵ (respectively viscosity and η) alone. Their expressions are then determined by dimensional analysis. We consider the Littlewood-Paley decomposition of the velocity associated to a mollifier $\phi(\xi)$. This is a smooth function in R^d that is non-increasing, smooth, radially symmetric, satisfying $\phi(\xi) = 1$ for $|\xi| \leq \frac{5}{8}$, $\phi(\xi) = 0$ for $|\xi| \geq \frac{3}{4}$. One sets $\psi_{(0)}(\xi) = \phi(\frac{\xi}{2}) - \phi(\xi)$, $\psi_{(m)}(\xi) = \psi_{(0)}(2^{-m}\xi)$ and

$$\phi(\xi) = \int_{\mathbf{R}^d} e^{-i(\xi \cdot z)} \Phi(z) dz,$$

$$\psi_{(m)}(\xi) = \int_{\mathbf{R}^d} e^{-i(\xi \dot{z})} \Psi_{(m)}(z) dz.$$

The Littlewood-Paley decomposition is

$$u(x,t) = u_{(-\infty)}(x,t) + \sum_{m=0}^{\infty} u_{(m)}(x,t)$$

where

$$u_{(-\infty)}(x,t) = L^{-d} \int_{\mathbf{R}^d} \Phi\left(\frac{y}{L}\right) u(x-y,t) dy,$$

$$u_{(m)}(x,t) = L^{-d} \int_{\mathbf{R}^d} \Psi_{(m)}\left(\frac{y}{L}\right) u(x-y,t) dy$$

and L is a length (the integral scale).

We define the Littlewood-Paley spectrum to be

$$E_{LP}(k) = k_m^{-1} \left\langle |u_{(m)}|^2 \right\rangle$$

for $k_{m-1} \leq k < k_m$, $m \geq 1$ with $k_m = 2^m L^{-1}$.

We start with the two dimensional Navier-Stokes equation

$$(\partial_t + u \cdot \nabla - \nu \Delta) \omega = f$$

with

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{y^{\perp}}{|y|^2} \omega(x-y,t) dy$$

One can prove ([24])

Theorem 9 *Assume that the source of vorticity in the Navier-Stokes equations has spectrum localized in the region of wave numbers $k \leq \frac{1}{L}$ for some $L > 0$. Then there exists a constant C such that the Littlewood-Paley energy spectrum of solutions of two dimensional forced Navier-Stokes equations obeys the bound*

$$E_{LP}(k) \leq Ck^{-3} \left\{ \tau^{-2} \left(\frac{k_d}{k} \right)^6 \right\}$$

for $k \geq L^{-1}$. Here $\tau^{-1} = <\|\nabla u\|_{L^\infty}>$.

The corresponding three-dimensional energy spectrum result requires a significant assumption:

$$\langle |\nabla u|^3 \rangle < \infty.$$

Denoting

$$\widehat{\epsilon} = \nu \left\{ \langle |\nabla u|^3 \rangle \right\}^{\frac{2}{3}}$$

$$\widehat{\eta} = \nu^{\frac{3}{4}} \left(\widehat{\epsilon} \right)^{-\frac{1}{4}}$$

$$\widehat{k}_d = \nu^{-\frac{3}{4}} (\widehat{\epsilon})^{\frac{1}{4}}$$

and setting

$$C_\psi = \int \int |\nabla \Psi_{(0)}(a)||a|^2 |\Psi_{(0)}(b)||b| da db.$$

we have ([25]):

Theorem 10 *Consider three-dimensional body forces that satisfy*

$$\widehat{f}(k) = 0$$

for all $|k| \geq \frac{C}{L}$ and some $C > 0$. Consider solutions of the three dimensional Navier-Stokes equation that satisfy $\widehat{\epsilon} < \infty$. Then

$$E_{LP}(k) \leq C_\psi \left(\widehat{\epsilon} \right)^{\frac{2}{3}} k^{-\frac{5}{3}} \left(\frac{k}{\widehat{k}_d} \right)^{-\frac{10}{3}}$$

holds for $|k| \geq \frac{C}{L}$.

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